Appendix B

The Bessel and Related Functions

We define the Bessel function of integer order k as

$$J_k(z) := (2\pi)^{-1} \int_{-\pi}^{\pi} d\theta \cos(z \sin \theta - k\theta). \tag{B.1}$$

This was Bessel's original definition in 1824 [see Watson (1922, Section 2.2 and the references therein)]. It leads to our first use of this function in Eqs. (5.48c)–(5.50) by the following steps involving trigonometric identities, considerations about the parity of the functions, and the invariance of (B.1) under $\theta \to \pm\theta + \alpha$:

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta \cos(z \sin \theta - k\theta) &= \int_{-\pi}^{\pi} d\theta \exp(iz \sin \theta) \exp(-ik\theta) \\ &= \int_{-\pi}^{\pi} d\theta \cos(z \sin \theta) \exp(-ik\theta).\end{aligned} \tag{B.2}$$

With the substitutions $\theta = x/2$, $k = 2(n - m)$, and $z = 2\tau(k/M)^{1/2}$ we obtain (5.48c).

The middle term in (B.2), with the substitution $t = \exp(i\theta)$, gives us (B.1) as a closed contour integral around the origin:

$$J_k(z) = (2\pi i)^{-1} \oint dt\, t^{-k-1} \exp[z(t - t^{-1})/2]. \tag{B.3}$$

It follows from here that the $J_k(z)$ are the *Laurent* series coefficients of the exponential function in the integrand, i.e.,

$$G_B(z, t) := \exp[z(t - t^{-1})/2] = \sum_{k \in \mathscr{Z}} J_k(z) t^k. \tag{B.4}$$

This is the Bessel *generating function*. For $t = i\exp(i\theta)$ and $z = pq$ we obtain (8.78). From (B.1) we can see that $J_k(z)$ is an analytic function of z in the neighborhood of $z = 0$. Its Taylor expansion can be found from (B.3) by the Taylor expansion of the exponential and the Cauchy integral (8.12) for $g(s) = s^p$:

$$J_k(z) = (2\pi i)^{-1} \oint dt \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{zt}{2}\right)^m \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-z}{2t}\right)^n t^{-k-1}$$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^n}{m!\,n!}\left(\frac{z}{2}\right)^{m+n} (2\pi i)^{-1} \oint dt\, t^{m-n-k-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+k)!}\left(\frac{z}{2}\right)^{2n+k}. \tag{B.5}$$

For k an integer it thus follows that

$$J_k(e^{i\pi m}z) = e^{ik\pi m}J_k(z). \tag{B.6}$$

The ratio test shows that this series converges for all finite z. Equation (B.5) can be generalized for complex values of the index by the gamma function:

$$J_k(z) := \sum_{n=0}^{\infty} (-1)^n [n!\,\Gamma(n+k+1)]^{-1}(z/2)^{2n+k}, \qquad k \in \mathscr{C}. \tag{B.7}$$

In particular, for $k = \pm\frac{1}{2}$, we find

$$J_{1/2}(z) = (2/\pi z)^{1/2} \sin z, \tag{B.8a}$$

$$J_{-1/2}(z) = (2/\pi z)^{1/2} \cos z. \tag{B.8b}$$

The Bessel function $J_k(z)$ has a countable infinity of simple real zeros for $z > 0$ and, at $z = 0$, a k-fold zero. The location of $j_{k,n}$, the nth zero of $J_k(z)$, is an increasing function of k. These zeros are transcendental numbers bounded from below by k, and as $n \to \infty$ their spacing increases monotonically, tending toward π. They interlace since $j_{k,n} < j_{k+1,n} < j_{k,n+1} < j_{k+1,n+1} < \cdots$. Table B.1 gives the first zeros of a few low-order Bessel functions.

Table B.1 Zeros of the Bessel Function j_{kn}

n \ k	0	1	2	3	4	5
1	2.40482	3.83171	5.13562	6.38016	7.58834	8.77148
2	5.52007	7.01559	8.41724	9.76102	11.06471	12.33860
3	8.65372	10.17347	11.61984	13.01520	14.37254	15.70017
4	11.79153	13.32369	14.79595	16.22347	17.61597	18.98013
5	14.93091	16.47063	17.95982	19.40942	20.82693	22.21780

A *Christoffel–Darboux* three-term recursion relation for Bessel functions can be obtained by differentiating the generating function (B.4) with respect to t:

$$\begin{aligned}\sum_{k\in\mathscr{Z}} kJ_k(z)t^{k-1} &= \partial G_B(z,t)/\partial t = (z/2 + z/2t^2)G_B(z,t)\\ &= \sum_{k\in\mathscr{Z}} z[J_k(z) + J_{k+2}(z)]t^k/2,\end{aligned} \tag{B.9}$$

where we have shifted the dummy sum index where necessary. Linear independence of the power functions then implies

$$J_{k+1}(z) - 2kz^{-1}J_k(z) + J_{k-1}(z) = 0. \tag{B.10}$$

Similarly, differentiating (B.4) with respect to z and rearranging terms by (B.9), we find the raising and lowering operators,

$$(k/z \mp d/dz)J_k(z) = J_{k\pm 1}(z), \tag{B.11}$$

which can also be seen to hold directly by the series expansion (B.7).

Equation (8.65) can be shown to hold for integer N (integer or half-integer μ) by noting that for $N = 2$ ($\mu = 0$) it coincides with (B.1) for $k = 0$. For $N = 3$ ($\mu = 1/2$) the integral is elementary and leads correctly to (B.8a). Last, the Bessel function as given by (8.65) can be seen to satisfy (B.11) by integration by parts.

Applying the raising and lowering operators (B.11) in either order to $J_k(z)$, we find that the Bessel function satisfies the second-order differential equation

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - k^2\right)J_k(z) = 0. \tag{B.12}$$

This is Bessel's differential equation. As is true for any (Fuchsian) equation, (B.12) has two independent solutions. They are $J_k(z)$ and $J_{-k}(z)$ for k *not* an integer; when k is an integer, these two functions are not linearly independent, but

$$J_{-k}(z) = (-1)^k J_k(z), \tag{B.13}$$

as can be ascertained from (B.7) by observing that the Γ-function in the denominator of $J_{-k}(z)$ has poles for $n = 0, 1, \ldots, k - 1$, so the sum actually starts from $n = k$. In the case of integer k, the *second* solution to (B.12), built for real k as

$$N_k(z) = [J_k(z)\cos(\pi k) - J_{-k}(z)]/\sin \pi k, \tag{B.14}$$

defines the *Neumann* function (also called Bessel of the *second* kind or Y-function). As k approaches integer values, $N_k(z)$ continues to be a well-defined function which can be found from L'Hospital's rule. Its explicit series expression can be found in the literature [see, for instance, Arfken (1966, Section 11.2), Whittaker and Watson (1903), and, of course, Watson's treatise (1922); the last contains a very complete account of these functions]. Bessel and Neumann functions have been plotted in Figs. B.1 and B.2.

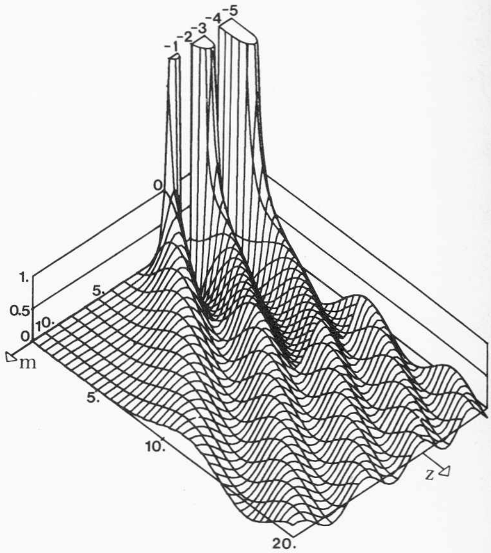

Fig. B.1. The Bessel function $J_m(z)$ for $m \in (-5, 10)$ and $z \in (0, 20)$. The function becomes infinite at $z = 0$ for all noninteger negative m. A closer grid details this region.

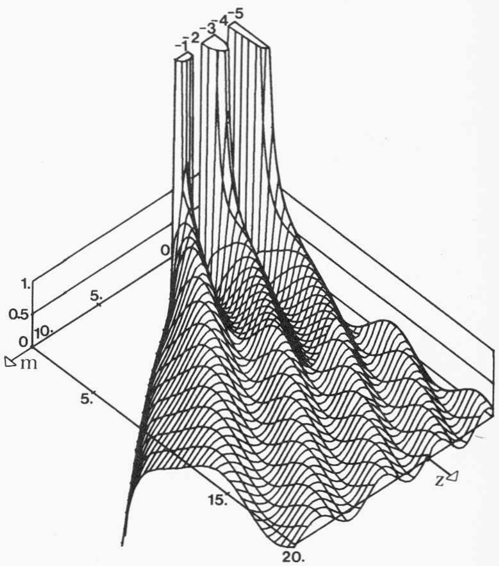

Fig. B.2. The Neumann function $N_m(z)$ for $m \in (-5, 10)$ and $z \in (0, 20)$. The point $z = 0$ is singular. A closer grid details the negative m region.

Some of the main properties of Bessel vs. Neumann functions are their behavior at the origin:

$$J_k(z) \underset{z\to 0}{\simeq} [2^k\Gamma(k+1)]^{-1}z^k, \qquad k \neq -1, -2, -3, \ldots, \tag{B.15a}$$

$$N_k(z) \underset{z\to 0}{\simeq} -2^k\pi^{-1}\Gamma(k)z^{-k}, \qquad k \neq 0, \qquad N_0(z) \underset{z\to 0}{\simeq} 2\pi^{-1}\ln z. \tag{B.15b}$$

Their asymptotic behavior can be shown to be

$$J_k(z) \underset{z\to\infty}{\simeq} (2/\pi z)^{1/2}\cos[z - \pi(k + 1/2)/2], \tag{B.16a}$$

$$N_k(z) \underset{z\to\infty}{\simeq} (2/\pi z)^{1/2}\sin[z - \pi(k + 1/2)/2]. \tag{B.16b}$$

[See Watson (1922, Chapter VII).] Both Bessel and Neumann functions satisfy the three-term and differential recursion equations (B.10) and (B.11). The properties of the zeros (simplicity, reality, spacing, and interlacing) are common to both functions.

The *modified* Bessel differential equation

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 - k^2\right)f(z) = 0 \tag{B.17}$$

has solutions $J_k(\pm iz)$. It is convenient to introduce the *modified* Bessel functions

$$I_k(z) := \begin{cases}\exp(i\pi k/2)J_k(\exp(i\pi/2)z), & -\pi < \arg z \leqslant \pi/2,\\ \exp(3i\pi k/2)J_k(\exp(-3i\pi/2)z), & \pi/2 < \arg z \leqslant \pi,\end{cases} \tag{B.18}$$

which have the series expansion

$$I_k(z) = \sum_{k=0}^{\infty} [n!\Gamma(n+k+1)]^{-1}(z/2)^{2n+k}. \tag{B.19}$$

[Compare with (B.7).] For k *not* an integer, independent solutions of (B.17) are provided by $I_k(z)$ and $I_{-k}(z)$. When k is an integer, $I_{-n}(z) = I_n(z)$. Suggesting analogy with Neumann functions, one defines the *Macdonald* function,

$$K_k(z) = (\pi/2)[I_{-k}(z) - I_k(z)]/\sin \pi k, \tag{B.20}$$

which is independent of $I_k(z)$ for all k. The limits and asymptotics of the I- and K-functions are

$$I_k(z) \underset{z\to 0}{\simeq} (z/2)^k/\Gamma(k+1), \qquad k \neq -1, -2, -3, \ldots, \tag{B.21a}$$

$$K_k(z) \underset{z\to 0}{\simeq} (z/2)^{-k}\Gamma(k)/2, \qquad k \neq 0; \qquad K_0(z) \underset{z\to 0}{\simeq} -\ln z, \tag{B.21b}$$

$$I_k(z) \underset{z\to\infty}{\simeq} (2\pi z)^{-1/2} \exp(z), \tag{B.22a}$$

$$K_k(z) \underset{z\to\infty}{\simeq} (\pi/2z)^{1/2} \exp(-z). \tag{B.22b}$$

For real z and positive k, both functions are free of zeros.

From the Bessel modified equation we shall obtain another important differential equation. Let $y := (3z/2)^{2/3}$ and write (B.17) in terms of it for $g(y) = f(z)$; then define $g(y) =: y^{-1/2}h(y)$. When $k = 1/3$, terms cancel, and we are left with *Airy*'s equation:

$$\left(\frac{d^2}{dy^2} - y\right)h(y) = 0. \tag{B.23}$$

Equation (7.61) is found from here by $y =: 2^{1/3}q$. It thus follows that $y^{1/2}I_{\pm 1/3}(2y^{3/2}/3)$ and $y^{1/2}K_{1/3}(2y^{3/2}/3)$ will be solutions to (B.23). Actually, one defines the *first* and *second Airy* functions:

$$\begin{aligned}\mathrm{Ai}(y) &:= \pi^{-1}(y/3)^{1/2}K_{1/3}(z)\\ &= y^{1/2}[I_{-1/3}(z) - I_{1/3}(z)]/3,\end{aligned} \tag{B.24a}$$

$$\mathrm{Ai}(-y) = y^{1/2}[J_{-1/3}(z) + J_{1/3}(z)], \tag{B.24b}$$

$$\mathrm{Bi}(y) := (y/3)^{1/2}[I_{-1/3}(z) + I_{1/3}(z)], \tag{B.24c}$$

$$\mathrm{Bi}(-y) = (y/3)^{1/2}[J_{-1/3}(z) - J_{1/3}(z)], \qquad z = 2y^{3/2}/3. \tag{B.24d}$$

These are plotted in Figure B.3. The Taylor expansions are of the form

$$\mathrm{Ai}(y) = c_1F_1(y) - c_2F_2(y), \tag{B.25a}$$

$$\mathrm{Bi}(y) = 3^{1/2}[c_1F_1(y) + c_2F_2(y)], \tag{B.25b}$$

where the constants are

$$c_1 = 3^{-2/3}/\Gamma(2/3) = 0.35502805\ldots = \mathrm{Ai}(0) = 3^{-1/2}\mathrm{Bi}(0), \tag{B.26a}$$

$$c_2 = 3^{-1/3}/\Gamma(1/3) = 0.25881940\ldots = -\mathrm{Ai}'(0) = 3^{-1/2}\mathrm{Bi}'(0) \tag{B.26b}$$

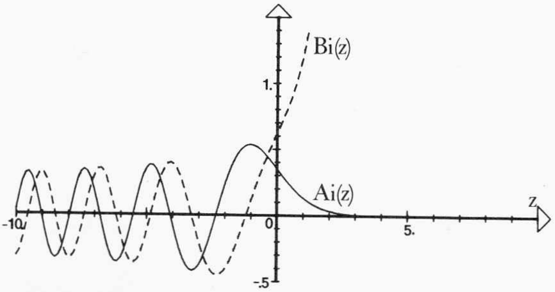

Fig. B.3. The Airy function of first and second kind.

and the functions are

$$F_n(y) = \sum_{m=0}^{\infty} 3^m (n/3)_m [\Gamma(3m + n)]^{-1} y^{3m+n-1}, \qquad n = 1, 2, \tag{B.27a}$$

where we have used *Pochhammer's symbol*

$$(a)_m := a(a + 1)(a + 2)\cdots(a + m - 1) = \Gamma(a + m)/\Gamma(a). \tag{B.27b}$$

For $y \to +\infty$, the asymptotics are

$$\mathrm{Ai}(y) \simeq 2^{-1}\pi^{-1/2}y^{-1/4}\exp(-2y^{3/2}/3), \tag{B.28a}$$

$$\mathrm{Ai}(-y) \simeq \pi^{-1/2}y^{-1/4}\sin(2y^{3/2}/3 + \pi/4), \tag{B.28b}$$

$$\mathrm{Bi}(y) \simeq \pi^{-1/2}y^{-1/4}\exp(2y^{3/2}/3), \tag{B.29a}$$

$$\mathrm{Bi}(-y) \simeq \pi^{-1/2}y^{-1/4}\cos(2y^{3/2}/3 + \pi/4). \tag{B.29b}$$

The integral expression (7.64), which we asserted represents the Airy function, can be put in terms of the usual and modified Bessel functions as in (B.24). The process is rather involved, so we refer the interested reader to the book by Watson (1922, Section 6.4). References to Airy's original "rainbow" equation and the solutions by Stokes and Hardy appear there.

Further properties and tables for the Bessel and related functions can be found in Abramowitz and Stegun (1964, Chapters 9 and 10), while integrals of Bessel functions—as for the Green's functions in Section 5.3—and Struve functions, mentioned in Section 8.5, occupy Chapters 11 and 12 of Abramowitz and Stegun. Further references and tables have been given in Sections 5.2, 5.3, and 6.4.